WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS METHODS
Differentiation, applications and anti-differentiation
Test 2

Name: Solutions. Marks: /50

Calculator Free (24 marks)

1. [3 marks]

Using calculus techniques, find two numbers whose difference is 32 and whose product is a minimum.

Let the numbers be x & y $x > y$

$x - y = 32$

$P = xy$ to be a min.

$P(x) = x(x-32)$ ✓

$= x^2 - 32x$

$P'(x) = 2x - 32 = 0 \Rightarrow x = 16$ ✓

$\therefore y = -16$

$P''(x) = 2 > 0 \Rightarrow$ min. ✓

[3]

2. [4 marks]

The displacement for an object is given by $x = \dfrac{2t-5}{3t+1}$, where x is in metres and t is in seconds. Find the equations for velocity and acceleration.

use of quotient rule ✓

$v = \dot{x} = \dfrac{2(3t+1) - 3(2t-5)}{(3t+1)^2} = \dfrac{17}{(3t+1)^2}$ ✓

$v = 17(3t+1)^{-2}$

use of chain rule ✓

$a = \dot{v} = \ddot{x} = -34(3t+1)^{-3} \times 3$

$= \dfrac{-102}{(3t+1)^3}$ ✓

[4]

3. [6 marks]

a) Find the coordinates of all stationary points on the curve $y = (2x+1)(x-2)^4$.

$$\frac{dy}{dx} = (2x+1) \cdot 4(x-2)^3 + 2(x-2)^4$$

$$= 2(x-2)^3 \left[2(2x+1) + (x-2) \right] \quad ✓$$

$$= 10x(x-2)^3 \quad = 0 \text{ for stat. points} \quad ✓$$

$\therefore \; x = 0 \Rightarrow y = 16 \qquad (0, 16) \quad ✓\tfrac{1}{2}$

$x = 2 \Rightarrow y = 0 \qquad (2, 0) \quad ✓\tfrac{1}{2}$

[3]

b) Sketch the curve, identifying the point of inflection *(x-value sufficient)*.

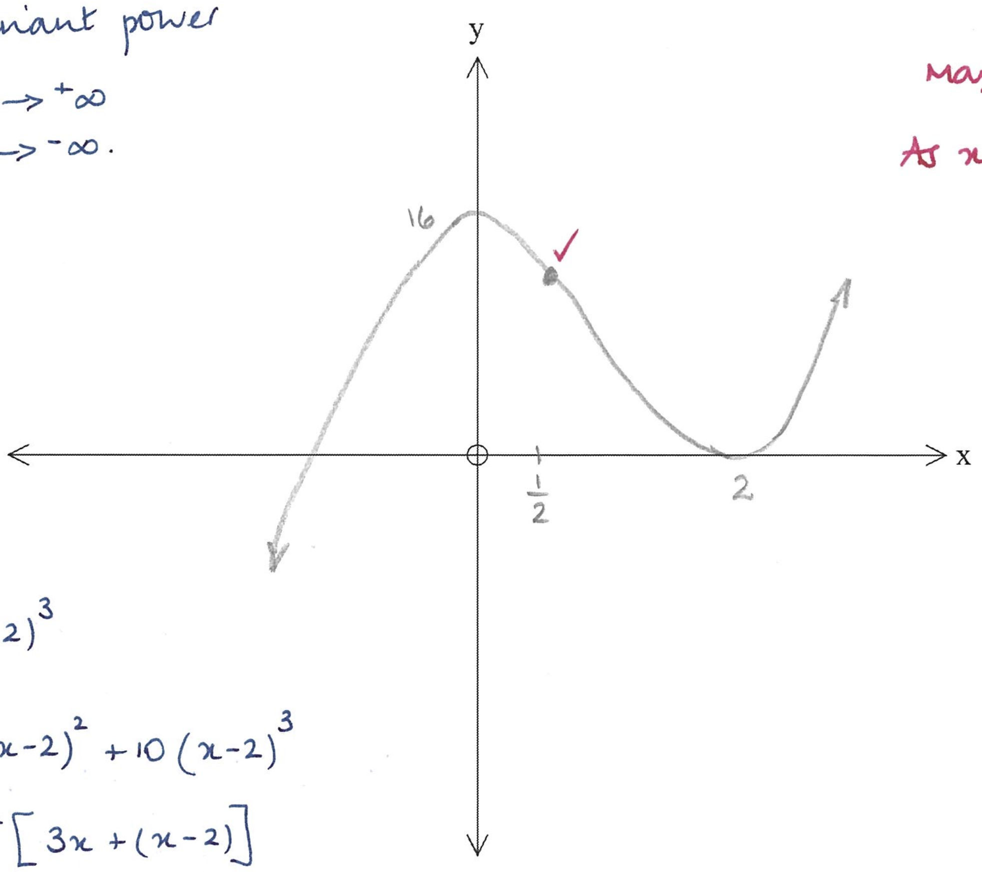
x^5 is the dominant power

As $x \to {}^+\infty \quad y \to {}^+\infty$

$x \to {}^-\infty \quad y \to {}^-\infty$.

Max/min ✓

As $x \to \pm\infty$ ✓



$$\frac{dy}{dx} = 10x(x-2)^3$$

$$\frac{d^2y}{dx^2} = 10x \cdot 3(x-2)^2 + 10(x-2)^3$$

$$= 10(x-2)^2 \left[3x + (x-2) \right]$$

$$= 10(x-2)^2(4x-2)$$

$x = 2 \qquad x = \frac{1}{2}$

[3]

4. [3 marks]

Given that $y = \sqrt[3]{x}$, use $x = 27$ and the incremental formula $\delta y \approx \frac{dy}{dx} \times \delta x$ to determine an approximate value for $\sqrt[3]{29}$.

$x = 27 \Rightarrow y = 3$ ✓

$\delta x = 2$

$y = x^{1/3}$

$\delta y \approx \frac{dy}{dx} \times \delta x$

$\approx \frac{1}{3} x^{-2/3} \times \delta x$ ✓

$\sqrt[3]{29} = y + \delta y$

$= 3 + \frac{1}{3(\sqrt[3]{27})^2} \times 2$

$= 3 + \frac{1}{3.3^2} \times 2$

$= 3\frac{2}{27}$ ✓

[3]

5. [2 marks]

Given that $f'(x) = 3x^3 - 3x^2$ and $f(2) = 7$, find $f(x)$.

$f(x) = \frac{3}{4}x^4 - x^3 + c$ ✓

$7 = \frac{3}{4}.16 - 8 + c \Rightarrow c = 3$

$f(x) = \frac{3}{4}x^4 - x^3 + 3.$ ✓

[2]

1mk each
(-1 if not "+c").

6. [6 marks]

Find the **antiderivative** of each of the following:

a) $2x^4$ $\longrightarrow$ $\dfrac{2x^5}{5} + c$

b) $\dfrac{x^3}{5}$ $\longrightarrow$ $\dfrac{x^4}{20} + c$

c) $\dfrac{4}{x^2}$ $\left(= 4x^{-2}\right)$ $\longrightarrow$ $\dfrac{4x^{-1}}{-1} + c = -\dfrac{4}{x} + c$

d) e^{5x} $\longrightarrow$ $\dfrac{e^{5x}}{5} + c$

e) $6e^{\frac{x}{3}}$ $\longrightarrow$ $18e^{x/3} + c$.

f) $\sqrt{2x-5}$ $\left(= (2x-5)^{1/2}\right)$ $\longrightarrow$ $\dfrac{(2x-5)^{3/2}}{\frac{3}{2} \times 2} + c = \dfrac{(2x-5)^{3/2}}{3} + c$.

End of Part A

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Calculator Section **(26 marks)**

7. [4 marks]

The population P of fish in a certain lake was studied over time, and at the start the number of fish was 2500.

a) During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

The population is decreasing

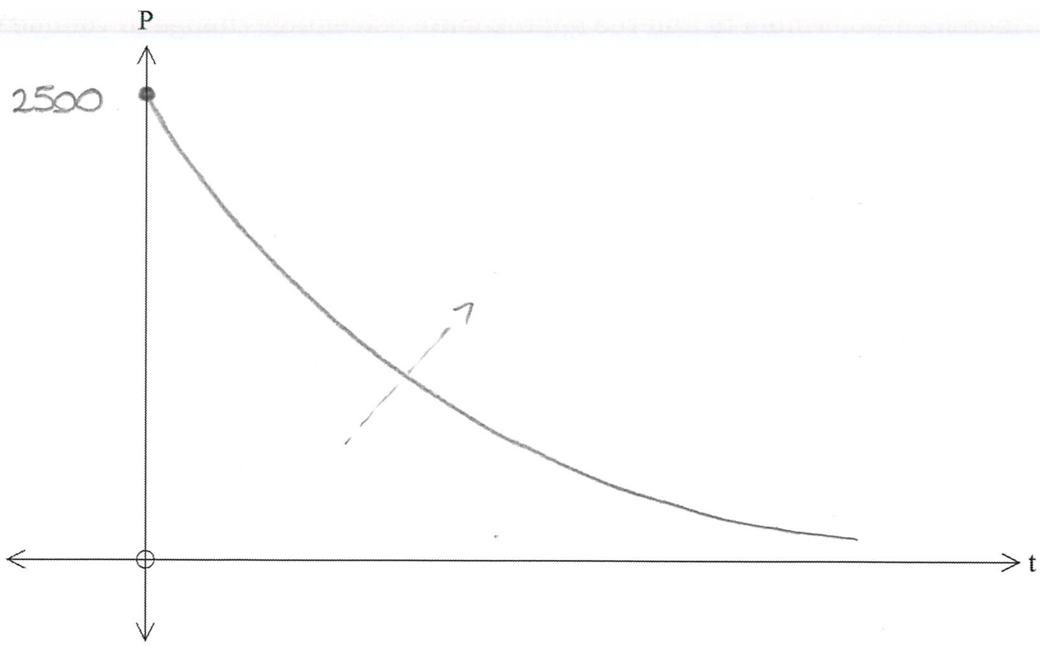
[1]

b) If, at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate?

Shape is concave up $\Rightarrow$ population is decreasing without reaching a min.

[1]

c) Sketch the graph of the population P against t.

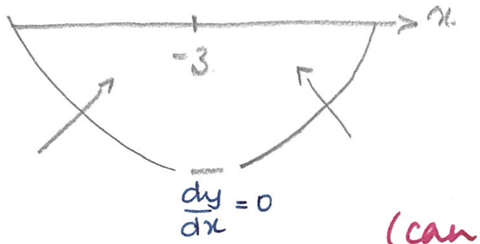


[2]

8. [2 marks]

For a certain curve, the derivative is zero when $x=-3$. Also $f''(-3)=0$ and $f''(x)>0$ either side of $x=-3$. Explain what kind of point is at $x=-3$.

concave up. ✓



$\Rightarrow$ it's a minimum. ✓

(can be above/below x-axis)

[2]

9. [5 marks]

a) At the Blackstrap Molasses factory, the profit in dollars depends on the amount (x kg) of molasses according to the formula $P=-x^3+69x^2+5040x+580$. What is the marginal profit after the 50th kg is sold?

$$\delta P \approx \left.\frac{dP}{dx}\right|_{x=50} \times \delta x \qquad \delta x = 1. \checkmark$$

$$\approx -3x^2+138x+5040\Big|_{x=50} = \$4440 \checkmark$$

(Also accept $\delta P = \$4275$ if stated $x=51$).

[2]

b) A spherical balloon is subjected to heat, causing it to expand uniformly. Use the incremental formula to find the approximate percentage change in volume when the diameter increases by 4%.

either

$\Rightarrow$ radius also increases by 4% ✓

$$V=\frac{4}{3}\pi r^3 \qquad \frac{\delta V}{V} \approx \frac{4\pi r^2 \times \delta r}{\frac{4}{3}\pi r^3} \checkmark = 3\frac{\delta r}{r} = 3\times 4\% = 12\% \checkmark$$

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or

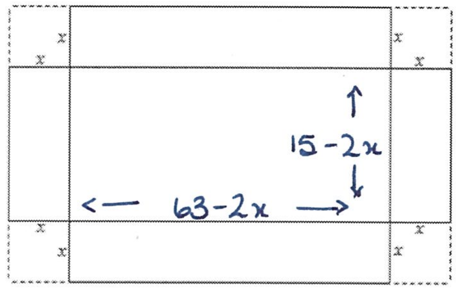
$$r=\frac{d}{2} \Rightarrow V=\frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{\pi d^3}{6} \checkmark$$

$$\frac{\delta V}{V} \approx \frac{\frac{\pi d^2}{2}\times \delta d}{\frac{\pi d^3}{6}} \checkmark = 3\frac{\delta d}{d} = 3\times 4\% = 12\%. \checkmark$$

[3]

10. [4 marks]

A box is made by cutting square corners out of a rectangular piece of tin and folding the sides up. If the original piece of tin measures 63 cm by 15 cm, and the squares have side length x cm, find using calculus techniques the volume of the box formed and justify that the volume is a maximum.



$V = x(63-2x)(15-2x)$ ✓

$V'(x) = 12x^2 - 312x + 945 = 0$ for stat points

$x = 3.5$ or ~~$x = 22.5$~~ not possible. ✓

$V''(x)\big|_{x=3.5} = -228 < 0 \Rightarrow$ max. ✓

$V(3.5) = 1568 \text{ cm}^3$. ✓

11. [11 marks]

A particle is initially at an origin O. It is then projected away from O and moves in a straight line such that its displacement from O, t seconds later is x metres where $x = t^3 - 6t^2 + 9t$.

$x(0) = 0$

Determine:

a) the initial speed of projection.

$\dot{x} = 3t^2 - 12t + 9$ ✓

$\dot{x}(0) = 9\text{m/s}$. ✓

[2]

b) when the particle is at rest and how far it is from the origin at these times.

$\dot{x} = 0$ ✓

$3t^2 - 12t + 9 = 0$

$3(t-1)(t-3) = 0$ ✓

$t = 1 \qquad t = 3$

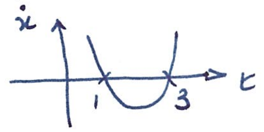
$x(1) = 1 - 6 + 9 = 4\text{m}$ ✓

$x(3) = 27 - 54 + 27 = 0\text{m}$ ✓

[4]

c) when the particle is moving in a positive direction.

ie when $\dot{x}(t) > 0$



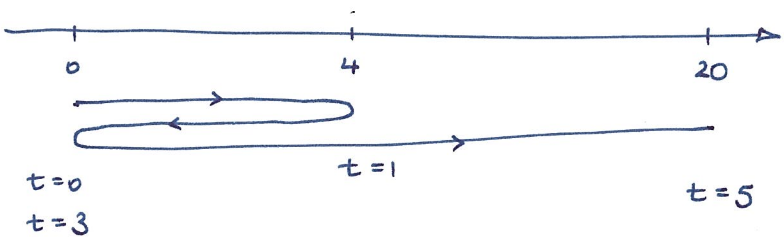
✓ ✓

$0 \leq t < 1$ or $t > 3$

[2]

d) the total distance travelled in the first 5 seconds.

either



✓

[3]

$x(5) = 20\text{m}$. ✓

$\therefore$ Distance travelled $= 4 + 4 + 20 = 28\text{m}$ ✓

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or

$$\int_0^5 |v|\,dt = \int_0^5 |3t^2 - 12t + 9|\,dt = 28\text{m}.$$

✓ ✓ ✓

End of Part B